

Persistence Diagrams as Diagrams: A Categorification of the Stability Theorem

Ulrich Bauer*

Michael Lesnick†

November 1, 2016

Abstract

Persistent homology, a central tool of topological data analysis, provides invariants of data called barcodes (also known as persistence diagrams). A barcode is simply a multiset of real intervals. Recent work of Edelsbrunner, Jabłoński, and Mrozek suggests an equivalent description of barcodes as functors $\mathbf{R} \rightarrow \mathbf{Mch}$, where \mathbf{R} is the poset category of real numbers and \mathbf{Mch} is the category whose objects are sets and whose morphisms are matchings (i.e., partial injective functions). Such functors form a category $\mathbf{Mch}^{\mathbf{R}}$ whose morphisms are the natural transformations. Thus, this interpretation of barcodes gives us a hitherto unstudied categorical structure on barcodes. The aim of this note is to show that this categorical structure leads to surprisingly simple reformulations of both the well-known stability theorem for persistent homology and a recent generalization called the induced matching theorem.

1 Introduction

The stability theorem is one of the main results of topological data analysis (TDA). It plays a key role in the statistical foundations of TDA [14], and is used to formulate theoretical guarantees for efficient algorithms to approximately compute persistent homology [6, 17].

The stability theorem is originally due to Cohen-Steiner et al., who presented a version of the theorem for the persistent homology of \mathbb{R} -valued functions [9]. Since then, the theorem has been revisited a number of times, leading to simpler proofs and more general formulations [1–5, 7, 8, 16]. In particular, Chazal et al. introduced the *algebraic stability theorem* [7], a useful and elegant algebraic generalization, and it was later observed that the (easy) converse to this result also holds [16]. Bubenik and Scott were the first to explore the category-theoretic aspects of the stability theorem, rephrasing some of the key definitions in terms of functors and natural transformations [4].

Letting \mathbf{vect} denote the category of finite dimensional vector spaces over a fixed field K , a *pointwise finite dimensional (p.f.d.) persistence module* is an object of the functor category $\mathbf{vect}^{\mathbf{R}}$. The structure theorem for p.f.d. persistence modules [10] tells us that the isomorphism type of a p.f.d. persistence module M is completely described by a barcode $\mathcal{B}(M)$; this barcode specifies the decomposition of M into indecomposable summands. The algebraic stability theorem, together with its converse, specifies in what sense similar persistence modules have similar barcodes.

In [1], the authors of the present note introduced the *induced matching theorem*, an extension of the algebraic stability theorem to a general result about morphisms of persistence modules, with a new, more

*Technical University of Munich, Germany. mail@ulrich-bauer.org

†Princeton University, NJ, USA. mlesnick@princeton.edu

direct proof. This note is intended as a follow-up to [1]; though it can be read independently, some familiarity with the earlier paper may be helpful for context.

The induced matching theorem can be viewed as a categorification of the stability theorem; see Remark 1.2 below. To formulate and prove the induced matching theorem, in [1] we considered the category whose objects are barcodes and whose morphisms are arbitrary matchings. In the present note, we introduce a different category of barcodes, denoted by **Barc**, with the same collection of objects but a smaller collection of morphisms and a slightly different composition operation, such that there exists an equivalence of categories $E : \mathbf{Barc} \rightarrow \mathbf{Mch}^{\mathbf{R}}$ extending the correspondence between barcodes and functors $\mathbf{R} \rightarrow \mathbf{Mch}$ given by Edelsbrunner, Jabłonski, and Mrozek [13]. We use this latter category of barcodes to further develop the categorical viewpoint on the stability theorem: We give simple reformulations of both the induced matching and algebraic stability theorems, which make clear for the first time that both results can be understood as the preservation of certain categorical structure. We first state our reformulation of the induced matching theorem. For \mathbf{A} a category with zero object, we will say that a diagram $M : \mathbf{R} \rightarrow \mathbf{A}$ is δ -trivial if for all $t \in \mathbb{R}$, the (categorical) image of the internal morphism $M_{t,t+\delta} : M_t \rightarrow M_{t+\delta}$ exists and is trivial. The empty set is the zero object in \mathbf{Mch} ; we say a barcode C is δ -trivial if $E(C)$ is δ -trivial. Using the definition of the equivalence E given below in Section 2, it is not hard to check that C is δ -trivial if and only if each interval of C is contained in some half-open interval of length δ . Similarly, a persistence module M is δ -trivial if and only if $\mathcal{B}(M)$ is δ -trivial.

Theorem 1.1 (Categorical Formulation of the Induced Matching Theorem). *A morphism $f : M \rightarrow N$ of p.f.d. persistence modules induces a morphism of barcodes $\mathcal{X}(f) : \mathcal{B}(M) \rightarrow \mathcal{B}(N)$ in **Barc** such that for $\delta \geq 0$,*

- (i) *if f has δ -trivial kernel, then so does $\mathcal{X}(f)$, and*
- (ii) *if f has δ -trivial cokernel, then so does $\mathcal{X}(f)$.*

We refer the reader to [1] for the definition of the map $f \mapsto \mathcal{X}(f)$.

Remark 1.2. While the map $f \mapsto \mathcal{X}(f)$ in the statement of Theorem 1.1 is not functorial, its restriction to the monomorphisms (epimorphisms) of p.f.d. persistence modules yields a functor. On the other hand, it is impossible to extend the map $M \mapsto \mathcal{B}(M)$ to a functor [1, Proposition 5.10]. The proof of the induced matching theorem relies on this partial functoriality in an essential way: We first establish the result for f a monomorphism by an appeal to functoriality. The full induced matching theorem then follows readily from this special case; in brief, we obtain the result for a general morphism f by considering the epi-mono decomposition of f and appealing to duality. See [1] for the details.

Remark 1.3. A morphism f in $\mathbf{vect}^{\mathbf{R}}$ is a monomorphism (epimorphism) if and only if f has a trivial kernel (respectively, cokernel), and it can be checked that the same is true as well for a morphism f in **Barc**. In view of this, taking $\delta = 0$ in Theorem 1.1 gives us that the map $f \mapsto \mathcal{X}(f)$ preserves monomorphisms and epimorphisms. This observation is equivalent to the structure theorem for submodules and quotients of persistence modules given in [1, Theorem 4.2].

We next turn to our reformulation of the algebraic stability theorem. We use the categorical structure on barcodes to state the theorem purely in terms of *interleavings* of \mathbf{R} -indexed diagrams, without introducing the definition of bottleneck distance on barcodes, as is usually done. Interleavings and the interleaving distance d_I can be defined on \mathbf{R} -indexed diagrams taking values in an arbitrary category; see Section 4 for a review of the definitions. We thus may define a δ -interleaving between barcodes C and \mathcal{D} (without

any reference to persistence modules) to be a δ -interleaving between the diagrams $E(C)$ and $E(\mathcal{D})$, and define $d_I(C, \mathcal{D}) := d_I(E(C), E(\mathcal{D}))$. We give a more direct description of interleavings between barcodes in Section 4.

Our Proposition 4.1 establishes that this distance d_I on barcodes is equal to the usual bottleneck distance on barcodes; in fact, we give a slightly sharper statement. From Proposition 4.1 and the usual forward and converse algebraic stability results, as stated in [1], we immediately obtain the following:

Theorem 1.4 (Categorical Formulation of Algebraic Stability). *Two p.f.d. persistence modules M and N are δ -interleaved if and only if their barcodes $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are δ -interleaved. In particular,*

$$d_I(M, N) = d_I(\mathcal{B}(M), \mathcal{B}(N)).$$

2 Barcodes as diagrams

In this section, we define the category **Barc** of barcodes. Then, building on ideas introduced in [13], we define the functor $E : \mathbf{Barc} \rightarrow \mathbf{Mch}^{\mathbf{R}}$ and observe that E is an equivalence.

Matchings First, we review some basic properties of the category **Mch** having sets as objects and matchings (partial injective functions) as morphisms. We will use the symbol \rightarrow to denote a matching. **Mch** is a subcategory of the category with sets as objects and relations as morphisms. The composition $\tau \circ \sigma : S \rightarrow U$ of two matchings $\sigma : S \rightarrow T$ and $\tau : T \rightarrow U$ is thus defined as

$$\tau \circ \sigma = \{(s, u) \mid (s, t) \in \sigma, (t, u) \in \tau \text{ for some } t \in T\}.$$

The monomorphisms in **Mch** are the injections, while the epimorphisms are the coinjections, i.e., matchings which match each element of the target. The kernel and cokernel of a morphism in **Mch** consist of the unmatched elements of the source and target, respectively, together with the canonical (co)injections. Similarly, the image and coimage consist of the matched elements.

Properties of Mch and Mch^R We next record several interesting properties of **Mch** and **Mch^R** in order to provide some context for our categorical study of barcodes. However, these properties will not be needed in what follows, and the reader may skip this part without loss of continuity. For more on the definitions introduced here, see [15].

Mch is an *inverse category*, i.e., every morphism $\sigma : S \rightarrow T$ has a generalized inverse $\tau : T \rightarrow S$ satisfying $\sigma \circ \tau \circ \sigma = \sigma$ and $\tau \circ \sigma \circ \tau = \tau$. An inverse category C is self-dual, i.e., C is equivalent to C^{op} ; in fact, C and C^{op} are isomorphic. Every inverse category is a *dagger category* (also known as an involutive category), i.e., a category C with an involutive functor $\dagger : C \rightarrow C^{\text{op}}$ that is the identity on objects. Here *involutive* means that $\sigma^{\dagger\dagger} = \sigma$ for all morphisms σ . (Some references, such as [18], require additionally that an involutive category comes equipped with a partial order on each hom-set $\text{hom}(a, b)$, and that these partial orders are compatible with composition. **Mch** also satisfies these additional conditions.)

Moreover, **Mch** is a *Puppe-exact category*: it has a zero object, it has all kernels and cokernels, every mono is a kernel, every epi is a cokernel, and every morphism has an epi-mono factorization. It has been shown in [15] that significant portions of homological algebra can be developed for Puppe-exact categories. Every Abelian category is Puppe-exact, but the converse is not true. In particular, **Mch** is not Abelian and not even additive; it does not have all binary (co)products.

The diagram category **Mch^R** inherits some of the properties of the category **Mch**. In particular, **Mch^R** is Puppe-exact. Moreover, **Mch^R** is self-dual, although not a dagger category, as the opposite category $(\mathbf{Mch}^{\mathbf{R}})^{\text{op}} \cong \mathbf{Mch}^{(\mathbf{R}^{\text{op}})}$ is equivalent to **Mch^R**, but only through an involutive functor that is not the identity on objects. The self-duality of **Mch** and **Mch^R** have been used in [1].

Barcodes Generalizing the definition in [1], we say a *multiset representation* consists of sets S and X , called the *base set* and the *indexing set* respectively, together with a subset $T \subseteq S \times X$ such that for each $s \in S$ there exists an index $x \in X$ with $(s, x) \in T$. For $s \in S$, the *multiplicity* of s in T is the cardinality of $\{x \in X \mid (s, x) \in T\}$. In [1], we considered a more restrictive definition of a multiset representation, where the indexing set was required to be a prefix of the natural numbers, but this more general definition will be convenient here.

Let T and T' be multiset representations with the same indexing set S and respective base sets X and X' . We say T' *reindexes* T , and write $T \cong T'$, if there exists a bijection $f : T \rightarrow T'$ such that for all $(s, x) \in T$, $f(s, x) = (s, x')$ for some $x' \in X'$. Note that \cong is an equivalence relation on multiset representations.

A *barcode* is a multiset representation whose base set consists of intervals in \mathbb{R} . In working with barcodes, we will often abuse notation slightly by suppressing the indexing set, and writing an element (s, x) of a barcode simply as s .

Category Structure on Barcodes For intervals $I, J \subseteq \mathbb{R}$, we say that I *bounds* J *above* if for all $s \in J$ there exists $t \in I$ with $s \leq t$. Symmetrically, we say that J *bounds* I *below* if for all $t \in I$ there exists $s \in J$ with $s \leq t$. We say that I *overlaps* J *above* if each of the following three conditions hold:

- $I \cap J \neq \emptyset$,
- I bounds J above, and
- J bounds I below.

For example, $[1, 3)$ overlaps $[0, 2)$ above, but $[0, 3)$ does not overlap $[1, 2)$ above. We define an *overlap matching* between barcodes \mathcal{C} and \mathcal{D} to be a matching $\sigma : \mathcal{C} \rightarrow \mathcal{D}$ such that if $\sigma(I) = J$, then I overlaps J above.

Note that if $\sigma : \mathcal{B} \rightarrow \mathcal{C}$ and $\tau : \mathcal{C} \rightarrow \mathcal{D}$ are both overlap matchings, then the composition $\tau \circ \sigma$ in **Mch** is not necessarily an overlap matching; for intervals I, J, K such that I overlaps J above, and J overlaps K above, it may be that $I \cap K = \emptyset$, so that I does not overlap K above.

We thus define the *overlap composition* $\tau \bullet \sigma$ of overlap matchings σ and τ as the matching

$$\tau \bullet \sigma = \{(I, K) \in \tau \circ \sigma \mid I \text{ overlaps } K \text{ above.}\}$$

It is easy to check that with this new definition of composition, the barcodes and overlap matchings form a category, which we denote as **Barc**. Note that two barcodes are isomorphic in **Barc** if and only if one reindexes the other. Note also that the empty barcode is the zero object in **Barc**.

Functor from Barcodes to Diagrams We now define the equivalence $E : \mathbf{Barc} \rightarrow \mathbf{Mch}^{\mathbb{R}}$. For \mathcal{D} a barcode and $t \in \mathbb{R}$, we let

$$E(\mathcal{D})_t := \{I \in \mathcal{D} \mid t \in I\},$$

and for each $s \leq t$ we define the internal matching $E(\mathcal{D})_{s,t} : E(\mathcal{D})_s \rightarrow E(\mathcal{D})_t$ to be the restriction of the diagonal of $\mathcal{D} \times \mathcal{D}$ to $E(\mathcal{D})_s \cap E(\mathcal{D})_t$, i.e.,

$$E(\mathcal{D})_{s,t} := \{(I, I) \mid I \in \mathcal{D}, s, t \in I\}.$$

We define the action of E on morphisms in **Barc** in the obvious way: for $\sigma : \mathcal{C} \rightarrow \mathcal{D}$ an overlap matching and $t \in \mathbb{R}$, we let $E(\sigma)_t : E(\mathcal{C})_t \rightarrow E(\mathcal{D})_t$ be the restriction of σ to pairs of intervals both containing t , i.e.,

$$E(\sigma)_t := \{(I, J) \in \sigma \mid t \in I \cap J\}.$$

It is straightforward to check that E is indeed a functor.

Functor from Diagrams to Barcodes To see that E is an equivalence, we next define a functor $F : \mathbf{Mch}^{\mathbf{R}} \rightarrow \mathbf{Barc}$ such that E and F are inverses (up to natural isomorphism).

For $D : \mathbf{R} \rightarrow \mathbf{Mch}$, let

$$\mathcal{E}(D) := \left(\bigcup_{t \in \mathbf{R}} \{t\} \times D_t \right) / \sim$$

where $(t, x) \sim (u, y)$ iff $(x, y) \in D(t, u)$ or $(y, x) \in D(u, t)$. The functoriality of D implies that the projection onto the first coordinate $(t, x) \mapsto t$ necessarily maps each equivalence class $Q \in \mathcal{E}(D)$ to an interval $I(Q) \subset \mathbf{R}$. We thus may define the barcode $F(D)$ by

$$F(D) := \{I(Q) \mid Q \in \mathcal{E}(D)\},$$

where we interpret the above expression as a multiset representation by taking the index of each $I(Q)$ to be Q . We take the action of F on morphisms to be the obvious one: for diagrams $C, D : \mathbf{R} \rightarrow \mathbf{Mch}$ and $\eta : C \rightarrow D$ a natural transformation (consisting of a family of matchings $\eta_t : C_t \rightarrow D_t$), we take $F(\eta) : F(C) \rightarrow F(D)$ to be the overlap matching given by

$$F(\eta) := \{(I(Q), I(R)) \mid Q \in \mathcal{E}(C), R \in \mathcal{E}(D), \text{ and } \exists t \in \mathbf{R}, (x, y) \in \eta_t \text{ with } (t, x) \in Q, (t, y) \in R\}.$$

It is easy to check that F is a functor and that E and F are indeed inverses up to natural isomorphism.

3 (Co)kernels of Barcodes and the Induced Matching Theorem

For $\sigma : C \rightarrow D$ an overlap matching of barcodes and $I \in C$, define

$$\ker(\sigma, I) = \begin{cases} I & \text{if } \sigma \text{ does not match } I, \\ I \setminus J & \text{if } \sigma(I) = J. \end{cases}$$

Hence, $\ker(\sigma, I)$ is either empty or an interval in \mathbf{R} . In the latter case, I and $\ker(\sigma, I)$ coincide above, in the sense that I overlaps $\ker(\sigma, I)$ above and $\ker(\sigma, I)$ also overlaps I above. Dually, for $J \in D$, we define

$$\text{coker}(\sigma, J) = \begin{cases} J & \text{if } \sigma \text{ does not match } J, \\ J \setminus I & \text{if } \sigma(I) = J. \end{cases}$$

Proposition 3.1. *For any morphism (i.e., overlap matching) $\sigma : C \rightarrow D$ in \mathbf{Barc} , the categorical kernel and cokernel of σ exist and are given by the barcodes*

$$\ker \sigma := \{\ker(\sigma, I) \mid I \in C\} \quad \text{and} \quad \text{coker } \sigma := \{\text{coker}(\sigma, J) \mid J \in D\},$$

together with the obvious matchings $\ker \sigma \rightarrow C$ and $D \rightarrow \text{coker } \sigma$.

Proof. As noted in Section 2, kernels and cokernels exist in \mathbf{Mch} ; for $\tau : S \rightarrow T$ a matching of sets, $\ker \tau$ (respectively, $\text{coker } \tau$) is simply the set of unmatched elements in S (respectively, T). It is a basic fact that for categories A, B where B has all (co)kernels, the functor category B^A has all (co)kernels, and these are given pointwise. Thus $\mathbf{Mch}^{\mathbf{R}}$ has all (co)kernels. (Indeed, as mentioned in Section 2, $\mathbf{Mch}^{\mathbf{R}}$ is a Puppe-exact category, which is a stronger statement.)

Given $D, D' : \mathbf{R} \rightarrow \mathbf{Mch}$, a natural transformation $\eta : D \rightarrow D'$, and $(x, y) \in D_{s,t}$ with $x \in \ker \eta_s$, we have that $y \in \ker \eta_t$. The result for kernels now follows from the definition of the equivalence $F : \mathbf{Mch}^{\mathbf{R}} \rightarrow \mathbf{Barc}$ and the description of kernels in $\mathbf{Mch}^{\mathbf{R}}$ given above. A dual argument gives the result for cokernels. \square

Let us now recall the statement of the induced matching theorem, as presented in [1]. We use slightly different notation than in [1], in order to avoid reintroducing the formalism of decorated endpoints used there. Given an interval $I \subset \mathbb{R}$ and $\delta \geq 0$, let $I(\delta) := \{t \mid t + \delta \in I\}$ be the interval obtained by shifting I downward by δ .

Theorem 3.2 (Induced Matching Theorem [1]). *A morphism $f : M \rightarrow N$ of p.f.d. persistence modules induces a matching of barcodes $X(f) : \mathcal{B}(M) \rightarrow \mathcal{B}(N)$ such that if $X(f)(I) = J$, then*

- (i) *I overlaps J above.*
- (ii) *If $\ker f$ is δ -trivial, then*
 - (a) *any interval of $\mathcal{B}(M)$ that is not δ -trivial is matched by $X(f)$, and*
 - (b) *J bounds $I(\delta)$ above.*
- (iii) *If $\operatorname{coker} f$ is δ -trivial, then*
 - (a) *any interval of $\mathcal{B}(N)$ that is not δ -trivial is matched by $X(f)$, and*
 - (b) *$I(\delta)$ bounds J below.*

Given Proposition 3.1, it is straightforward to verify that Theorem 1.1 is equivalent to Theorem 3.2, as claimed.

Example 3.3. Interestingly, the map $f \mapsto X(f)$ may strictly decrease the triviality of (co)kernels: we give an example of a morphism $f : M \rightarrow N$ such that $\operatorname{coker} f$ is not 2-trivial but $\operatorname{coker} X(f)$ is 2-trivial. Let C^J denote the interval persistence module [1] corresponding to an interval $J \subset \mathbb{R}$, and let

$$M = C^{[2,4)}, \quad N = C^{[0,4)} \oplus C^{[1,3)}, \quad \text{and} \quad f = (1 \ 1).$$

Then $\mathcal{B}(\operatorname{coker} f) = \{[0, 3), [1, 2)\}$ and $\operatorname{coker} X(f) = \{[0, 2), [1, 3)\}$.

4 Interleavings of Barcodes

The definition of interleavings of \mathbf{R} -indexed diagrams was introduced in [7], building on ideas in [9], and was first stated in categorical language in [4]. Though interleavings over more general indexing categories can be defined and are also of interest in TDA [5, 11, 12, 16], we focus here on the \mathbf{R} -indexed case.

Interleavings of \mathbf{R} -Indexed Diagrams Consider the translation $t \mapsto t + \delta$ of the real line by $\delta \geq 0$ as an endofunctor $S_\delta : \mathbf{R} \rightarrow \mathbf{R}$. For any category \mathbf{A} and diagram $M : \mathbf{R} \rightarrow \mathbf{A}$, we write $M(\delta) := M \circ S_\delta$. Thus, $M(\delta)$ is the diagram obtained by shifting each vector space and linear map in M downward by δ . An *interleaving* between two diagrams $M, N : \mathbf{R} \rightarrow \mathbf{A}$ is a pair of natural transformations $f : M \rightarrow N(\delta)$, $g : N \rightarrow M(\delta)$ such that $g_{t+\delta} \circ f_t = M_{t,t+2\delta}$ and $f_{t+\delta} \circ g_t = N_{t,t+2\delta}$ for all $t \in \mathbb{R}$. The interleaving distance on objects of $\mathbf{A}^{\mathbf{R}}$ is then given by

$$d_I(M, N) := \inf \{\delta \geq 0 \mid M \text{ and } N \text{ are } \delta\text{-interleaved.}\}$$

Interleavings in Barc As noted in Section 1, we can define a δ -interleaving between barcodes C and \mathcal{D} simply to be a δ -interleaving between $E(C)$ and $E(\mathcal{D})$. We next give a description of such interleavings directly in terms of the morphisms of **Barc**.

For C a barcode, let

$$C(\delta) := \{I(\delta) \mid I \in C\},$$

and let $\varphi^C(\delta) : C \rightarrow C(\delta)$ be the overlap matching given by

$$\varphi^C(\delta) := \{(I, I(\delta)) \mid I \text{ is not } \delta\text{-trivial}\}.$$

Given barcodes C and \mathcal{D} , we say a pair of overlap matchings

$$f : C \rightarrow \mathcal{D}(\delta), \quad g : \mathcal{D} \rightarrow C(\delta)$$

is a δ -interleaving if $g \bullet f = \varphi^C(2\delta)$, and $f \bullet g = \varphi^{\mathcal{D}}(2\delta)$. This is equivalent to the definition of interleavings of barcodes we have given via the functor E , in the sense that a pair of overlap matchings

$$f : C \rightarrow \mathcal{D}(\delta), \quad g : \mathcal{D} \rightarrow C(\delta)$$

is a δ -interleaving if and only if the pair $E(f), E(g)$ is a δ -interleaving in **Mch^R**.

Bottleneck Distance For $I \subset \mathbb{R}$ an interval and $\delta \geq 0$, let the interval $\text{Ex}^\delta(I)$ be given by

$$\text{Ex}^\delta(I) := \{t \in \mathbb{R} \mid \exists s \in I \text{ with } |s - t| \leq \delta\}.$$

We define a δ -matching between barcodes C and \mathcal{D} to be a (not necessarily overlap) matching $\sigma : C \rightarrow \mathcal{D}$ with the following two properties:

- (i) σ matches each interval in $C \cup \mathcal{D}$ that is not 2δ -trivial,
- (ii) if $\sigma(I) = J$, then $I \subset \text{Ex}^\delta(J)$ and $J \subset \text{Ex}^\delta(I)$.

We define the *bottleneck distance* d_b by taking

$$d_b(C, \mathcal{D}) := \inf \{\delta \geq 0 \mid \exists \text{ a } \delta\text{-matching between } C \text{ and } \mathcal{D}\}.$$

Interleaving Distance Equals Bottleneck Distance on Barcodes For \mathcal{D} any barcode, let $r^\delta : \mathcal{D}(\delta) \rightarrow \mathcal{D}$ be the obvious bijection.

Proposition 4.1. *An overlap matching of barcodes $f : C \rightarrow \mathcal{D}(\delta)$ is a δ -interleaving morphism if and only if $r^\delta \circ f$ is a δ -matching. In particular, for any barcodes C and \mathcal{D} ,*

$$d_I(C, \mathcal{D}) = d_B(C, \mathcal{D}).$$

Proof. It is easy to check that an overlap matching $f : C \rightarrow \mathcal{D}(\delta)$ is a δ -interleaving morphism if and only if f has 2δ -trivial kernel and cokernel. Moreover, an overlap matching $f : C \rightarrow \mathcal{D}(\delta)$ has 2δ -trivial kernel and cokernel if and only if $r^\delta \circ f$ is a δ -matching. \square

In its strong formulation for p.f.d. persistence modules [1], the algebraic stability theorem was phrased in terms of δ -matchings and the bottleneck distance as follows:

Theorem 4.2 (Forward and Converse Algebraic Stability [1, 7, 8, 16]). *Two p.f.d. persistence modules M and N are δ -interleaved if and only if there exists a δ -matching between $\mathcal{B}(M)$ and $\mathcal{B}(N)$. In particular,*

$$d_I(M, N) = d_B(\mathcal{B}(M), \mathcal{B}(N)).$$

Given Proposition 4.1, it is clear that Theorem 4.2 is equivalent to our new formulation of algebraic stability, Theorem 1.4.

5 Discussion

In this note, we have established some basic facts about the category $\mathbf{Barc} \cong \mathbf{Mch}^{\mathbf{R}}$ of barcodes and used these observations to give simple new formulations of the induced matching and algebraic stability theorems.

In fact, our definition of the category \mathbf{Barc} extends to barcodes indexed over arbitrary posets, as defined in [3], and many of the ideas presented here extend either to arbitrary posets or to \mathbf{R}^n -indexed barcodes for any n . In particular, Proposition 4.1 extends to \mathbf{R}^n -indexed barcodes, and this provides alternative language for expressing generalized algebraic stability results appearing in [2, 3]. While it remains to be seen what role the categorical viewpoint on barcodes might play in the further development of TDA theory, we hope that it might offer some perspective on the question of how algebraic stability ought to generalize to other settings.

As already mentioned, our new formulations of the algebraic stability and induced matching theorems make clear that both results can be interpreted as the preservation of some categorical structure as we pass from $\mathbf{vect}^{\mathbf{R}}$ to \mathbf{Barc} . Can more be said about how the passage from persistence modules to barcodes preserves categorical structure? As we have noted in Section 2, $\mathbf{Mch}^{\mathbf{R}}$ is a Puppe-exact category, and significant portions of homological algebra can be developed for such categories. We wonder whether our results can be understood as part of a larger story about how homological algebra in the Abelian category $\mathbf{vect}^{\mathbf{R}}$ relates to homological algebra in \mathbf{Barc} .

Acknowledgements

This research has been partially supported by the DFG Collaborative Research Center SFB/TRR 109 “Discretization in Geometry and Dynamics,” NIH grant T32MH065214, and an award from the J. Insley Blair Pyne Fund.

References

- [1] U. Bauer and M. Lesnick. [Induced matchings and the algebraic stability of persistence barcodes](#). *Journal of Computational Geometry*, 6(2):162–191, 2015.
- [2] H. B. Bjerkevik. [Stability of higher-dimensional interval decomposable persistence modules](#). Preprint, 2016. [arXiv:1609.02086](#).
- [3] M. B. Botnan and M. Lesnick. [Algebraic stability of zigzag persistence modules](#). Preprint, 2016. [arXiv:1604.00655](#).
- [4] P. Bubenik and J. A. Scott. [Categorification of persistent homology](#). *Discrete & Computational Geometry*, 51(3):600–627, 2014.
- [5] P. Bubenik, V. de Silva, and J. Scott. [Metrics for generalized persistence modules](#). *Foundations of Computational Mathematics*, 15(6):1501–1531, 2015.
- [6] N. J. Cavanna, M. Jahansseir, and D. R. Sheehy. [A geometric perspective on sparse filtrations](#). Preprint, 2015. [arXiv:1506.03797](#).
- [7] F. Chazal, D. C. Steiner, M. Glisse, L. J. Guibas, and S. Y. Oudot. [Proximity of persistence modules and their diagrams](#). In *Proceedings of the 25th Annual Symposium on Computational Geometry*, SCG ’09, pages 237–246. ACM, 2009.
- [8] F. Chazal, V. de Silva, M. Glisse, and S. Oudot. [The Structure and Stability of Persistence Modules](#). Springer-Briefs in Mathematics. Springer, 2016.

- [9] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. [Stability of persistence diagrams](#). *Discrete & Computational Geometry*, 37(1):103–120, 2007.
- [10] W. Crawley-Boevey. [Decomposition of pointwise finite-dimensional persistence modules](#). *Journal of Algebra and Its Applications*, 14(5):1550066+, 2015.
- [11] J. Curry. [Sheaves, Cosheaves and Applications](#). PhD thesis, University of Pennsylvania, 2014. [arXiv:1303.3255](#).
- [12] V. de Silva, E. Munch, and A. Patel. [Categorified Reeb graphs](#). 55(4):854–906, 2016.
- [13] H. Edelsbrunner, G. Jabłoński, and M. Mrozek. [The persistent homology of a self-map](#). *Foundations of Computational Mathematics*, 15(5):1213–1244, 2015.
- [14] B. T. Fasy, F. Lecci, A. Rinaldo, L. Wasserman, S. Balakrishnan, and A. Singh. [Confidence sets for persistence diagrams](#). *The Annals of Statistics*, 42(6):2301–2339, 2014.
- [15] M. Grandis. *Homological Algebra: The interplay of homology with distributive lattices and orthodox semi-groups*. World scientific, 2012.
- [16] M. Lesnick. [The theory of the interleaving distance on multidimensional persistence modules](#). *Foundations of Computational Mathematics*, 15(3):613–650, 2015.
- [17] D. R. Sheehy. [Linear-Size approximations to the Vietoris–Rips filtration](#). *Discrete & Computational Geometry*, 49(4):778–796, 2013.
- [18] E. W. Weisstein. *CRC concise encyclopedia of mathematics*. CRC press, 2002.